FORMAL RECORD 17

Gravity, Carried

On the Production of Weak-Field Gravity by Held Boundaries

Falling is carrying on.

Author: Stacy Gildenston
Affiliation: 3primitives.io
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v2.3 corrects wording for readability and precision. Conditions stated in the body now also appear where the results are summarised. No mathematics has changed.

Abstract. Formal Record 16 proved that held boundaries make spacetime: keepers scattered at random, each holding its own cell, ticking on its own clock and pushing the neighbours it touches, make at large scale the Minkowski spacetime of special relativity, in every frame. Its keepers ticked at one rate in a given flat space, and its scope excluded gravity.

This record lets the clocks run at different rates from place to place and the cells differ in size and shape, keeps every keeper where it is, and proves that, for clock rates and cells given in advance and with the amount of space at each place fixed in time, the order such a world makes is, at large scale, exactly the curved spacetime $g = -c_0^2 N^2\,dt^2 + h$ that those rates and cells describe: its first arrivals, its interval counts, its related pairs and its chains (Theorems B, W and O). In the tension world, density then follows extent by a theorem (Theorem LE).

Matter is a lump of held keepers, its mass σ_0 times its held count. A lump's own time is read from the order. A free lump carries only its last step, and exactly one continuation picks out no frame: straight on. In the order's own geometry that is the geodesic: for a lump that acts at its own grain (Condition G), steps smoothly (Condition S) and carries nothing beyond its last step, free fall is forced, not chosen; for a rule with chance in it, the mean step is forced (Theorem 9.4). Each keeper reads matter from its own past light cone, by a form of reading this record adopts (Definition 10.2). It carries forward the latest episode of every source, with the one weight no motion can detect, and in three space dimensions that reading moves at the push speed with nothing tuned (Proposition 10.1). Frame-freeness and Newton's law fix the coupling up to one ratio, and one premise, that a keeper's extent follows its pace, fixes that ratio at Einstein's value.

Given that premise and that reading, and where the order reads the clocks and cells the reading sets (Section 12), held edges then give gravity at first order exactly as Einstein's theory does, for static and steadily spinning sources and for gravitational waves: Newton's law, light bent by $4GM/(c_0^2 b)$, every lump falling alike, transverse gravitational waves at the push speed, and the Lense–Thirring dragging of orbits, with one constant, G, fixed by Newton's law (Theorem 11.1). Because keepers stay where they are and cell size is fixed in time, each field is held in the form with no shift and fixed cell size; the local rule by which they take up a wave's shape is named as a Target. The premise cannot be derived from the rest, which equally admit the scalar–tensor alternative (Proposition 10.4); the Cassini measurement fixes it to two parts in 10^5. Theorems B, W and O are proved for clock rates and cells given in advance; that they hold when the reading itself sets them is not proved here, and Theorem 11.1 is stated on that condition (Section 12).

In brief. Give the keepers of FR16 clocks that can run slow and cells that can shrink, and put a lump of held keepers among them. The pattern of cause and effect they make is curved spacetime, and the curvature is gravity: clocks near the lump run slow, light bends past it by Einstein's amount, everything falls toward it alike, a spinning lump drags orbits round, and ripples travel outward at the speed of light. Falling is not a pull. A free lump knows only its last step, and the only step that favours no direction is to carry straight on. The speed at which gravity travels is not tuned: it is the push speed. One premise, that cells shrink where clocks slow, sets how much light bends; it is the one place this record takes from nature what Einstein's field equations would otherwise supply, and experiment has measured it to two parts in a hundred thousand. The way each keeper reads matter, looking back along its own past light cone, is adopted, and one constant, G, is fixed by Newton's law. One step is open: the proof that the order makes curved spacetime covers clocks and cells set in advance, not yet those the reading sets (Section 12).

0. Dependencies and Scope

This record depends on the following prior works. No prior result is re-derived.

The Boundary Theorem (FR15). An edge is its keeper's own only when held from within and tested from without. Its condition H2, that a held boundary has extent, gives every length in Section 8 its scale, and Premise P its anchor.

Classical, Quantum, Consciousness (FR16). Supplies the clock-free conditions (C), (E) and (M), composites (Definition 5) and the causal order of held boundaries (Definitions 1 to 3, Axiom 1); the Minkowski limit for Poisson worlds and for cell worlds with W1 to W3 (Theorems 5.10 and 5.11, Lemmas 5.1 to 5.9, Appendix A); and the tension world (Definitions 7 to 10, Theorem 6.1, Appendix B). Every section here builds on it.

Scope. Sections 1 to 8 are mathematics about worlds of held cells whose clock rates, sizes and shapes vary slowly, with every keeper staying where it is and the amount of space at each place fixed in time. Sections 9 to 11 add matter as dust, its motion, and the reading by which keepers carry gravity, and derive gravity at first order in the field and in velocity. One premise is declared (Premise P, Section 10), one form of reading is adopted (Definition 10.2), and what makes a lump free is stated as two conditions (Conditions G and S, Section 9). Section 12 lists what is not covered, names one Target, places the record in the corpus's two tiers and states how it can fail. No governance result depends on this record. As in FR16, the limits are proved without rates of convergence.

1. The Varying World

Definition 1 (The varying world). *Fix $d \geq 2$, a spatial metric $h(x, t)$ and a clock rate $N(x, t)$, both smooth, with h between $\lambda_- I$ and $\lambda_+ I$ and N between N_- and N_+ (positive constants), and with $\det h$ independent of t. Fix a cell scale $\varepsilon > 0$. Keepers are the points of a Poisson process of intensity $\varepsilon^{-d}\sqrt{\det h(x)}$, and they do not move. Keeper k's clock is a Poisson process of rate $\varepsilon^{-1}N(x_k, t)$, independent of every other clock and of the keepers. At time t keeper k holds the cell $C_k(t) = \{z : d_t(z, x_k) \leq d_t(z, x_j)$ for every keeper $j\}$, where d_t is the distance of the metric $h(\cdot, t)$. Two keepers are neighbours at time t when their cells share a piece of dimension $d - 1$. A tick of keeper j at time s pushes the next tick of keeper k after s when j and k are neighbours at time s; ticks of one keeper are carried; $\lessdot$ is the transitive closure.*

This is FR16 Section 5's Setting with the metric's distance in place of the flat one, with each push read at the sender's tick (the sender's convention); with $h = \delta$ and $N = 1$ it is FR16's Setting, rescaled by ε. The world is static when h and N do not depend on t. Round cells are the case $h = \ell(x)^{-2}\delta$, with $\ell(x)$ the extent of the cells at x in units of ε.

Remark (why this definition). For ε small a cell lies within a few cells of its keeper, where h differs from h(x_k) by order ε, so C_k is, up to order ε, a cell shaped by the metric at its keeper's own place. Cells of a single metric fill space, meet only on their boundaries, and are star-shaped about their keepers along the metric's shortest paths once ε is below the scale on which h varies, so each is connected (Leibon and Letscher 2000; Boissonnat, Dyer and Ghosh 2018). One cell per unit h-volume on average says that every length scales with a keeper's extent (Section 8).

Definition 2 (the target spacetime). *$g = -c_0^2 N(x,t)^2\,dt^2 + h_{ij}(x,t)\,dx^i\,dx^j$, with c_0 the front speed of FR16 Lemma 5.5 for the Poisson world of intensity 1 in d dimensions. For a source point (x_0, t_0), $T^g(y)$ is the earliest time t at which a future-directed causal curve of g from (x_0, t_0) reaches (y, t). When g is static, $T^g(y) = t_0 + d(x_0, y)$, where $d(x, y)$ is the least of $\int |\gamma'|_h / (c_0 N)$ over paths from x to y (the optical distance).*

Definition 3 (the frozen world at (x_0, t_0)). *The same construction with h and N replaced by their constant values $h_0 = h(x_0, t_0)$ and $N_0 = N(x_0, t_0)$: keepers of intensity $\varepsilon^{-d}\sqrt{\det h_0}$, cells of the metric h_0, clocks of rate $\varepsilon^{-1} N_0$.*

Lemma 1.1 (a frozen world is FR16's world). *Let T be the positive square root of h_0 and $\Phi(x,t) = (\varepsilon^{-1} T(x - x_0),\ N_0 (t - t_0)/\varepsilon)$. Then Φ carries the frozen world to FR16 Section 5's Setting: keepers to a Poisson process of intensity 1, cells to their Voronoi cells, clocks to rate-1 clocks, and the order to FR16's order. Hence every lemma of FR16 Section 5 and Appendix A holds for the frozen world after Φ, its passage is $\rho_F(x,y) = (\varepsilon/N_0)\cdot\rho_1(\Phi x, \Phi y)$, and for fixed $x \neq y$, $\rho_F(x,y) \to |x - y|_{h_0} / (c_0 N_0)$ in probability and in mean as $\varepsilon \to 0$.*

Proof. A Poisson process of intensity μ carried by an affine map of determinant J is Poisson of intensity $\mu/|J|$, and here $\mu/|J| = \varepsilon^{-d}\sqrt{\det h_0}\,/(\varepsilon^{-d} \det T) = 1$. A cell of the metric h_0 is the set of points whose h_0-distance $|T(z - x_k)|$ to its keeper is least, which Φ carries to a Voronoi cell. A rate-$\varepsilon^{-1}N_0$ clock in t is a rate-1 clock in $N_0 t/\varepsilon$. The order is defined from neighbours and ticks alone, so Φ carries it. Passage scales with time, giving $\rho_F = (\varepsilon/N_0)\rho_1$; and $|\Phi x - \Phi y| = |T(x - y)|/\varepsilon \to \infty$, so Lemma 5.5 gives $\rho_1(\Phi x, \Phi y)\cdot\varepsilon/|T(x - y)| \to 1/c_0$ in probability and in mean. ∎

Units. Lengths and times below are physical: distances in the coordinates x, times in t. A cell is of size ε. A cube of side s means a cube of side sε, s cells across. Constants C, c, κ depend only on d, $\lambda_\pm$, $N_\pm$, the derivatives of h and N on the compact set considered, and on FR16's constants for the unit Poisson world, unless a dependence is shown.

2. Lemma M: Where Neighbours Can Change

A cell's neighbours are fixed by which keepers are nearest to the points around it. They can change under a small change of distance only where several keepers are almost equally near, that is, where the keepers sit in a nearly balanced arrangement. Lemma M makes this exact and shows such arrangements are rare.

Notation. Work in coordinates in which $h_0 = I$, by applying T. For d + 1 points σ, let $\nu(\sigma) = \mathrm{vol}(\sigma)/\mathrm{diam}(\sigma)^d$, its normalised volume; when $\nu(\sigma) > 0$, c_σ and r_σ are the centre and radius of the sphere through σ. Let $\Lambda = (\lambda_+/\lambda_-)^{1/2}$. For a locally finite point set P in general position, a region U, a length s and κ, γ in (0, 1), call U (κ, γ)-regular at scale s when:

(R1) every point within 4Λs of U has a point of P within s/(4Λ);

(R2) every d + 1 points of P within 6Λs of U whose diameter is at most 6Λs have $\nu \geq \kappa$;

(R3) for every such σ with $r_\sigma \leq 3\Lambda s$, no point q of P outside σ has $||q - c_\sigma| - r_\sigma| \leq \gamma\, r_\sigma$.

A function D(z, p) is τ-close at scale s when, for $|z - p| \leq 3\Lambda s$: $|D(z,p) - |z - p|| \leq \tau|z - p|$, $|\nabla_z D(z,p) - (z - p)/|z - p|| \leq \tau$ and $\|\nabla_z^2 D(z,p) - \nabla_z^2 |z - p|\| \leq \tau/|z - p|$; and $D(z,p) \geq |z - p|/\Lambda$ for all z and p. Its cells are $\{z : D(z,p) \leq D(z,q)$ for every q in P$\}$.

Lemma M. *(a) For a Poisson process of intensity between Λ_0^{-1} and Λ_0 and a set U of diameter at most s, (R1) fails with probability at most $C\exp(-c\,s^d)$, and (R2) and (R3) fail with probability at most $C(s)\kappa$ and $C(s)\gamma$. (b) There are c_1 and C_2, depending on d and Λ, such that if U is (κ,γ)-regular at scale s, $\tau \le c_1\kappa^{2(d+1)}$ and $C_2\tau\kappa^{-(d+1)} \le \gamma$, then for any two functions D and D' that are τ-close at scale s, every keeper whose cell meets U has the same neighbours in the cells of D, in the cells of D', and in the Voronoi cells of P.*

Proof. (a), (R1). This is the empty-ball bound of FR16 Lemma 5.3 and Proposition A.9: a point without a keeper within $s/(4\Lambda)$ lies near a ball of radius $s/(8\Lambda)$ empty of a process of intensity at least $\lambda_-^{(d/2)}\varepsilon^{-d}$ per unit physical volume, which in cell units has probability at most $\exp(-c\,s^d)$, and a fixed net of points covers the region.

(a), (R2). By the Mecke equation (Last and Penrose 2017) the expected number of $(d+1)$-tuples of P within $6\Lambda s$ of U with diameter at most $6\Lambda s$ and $\nu < \kappa$ is at most Λ_0^{d+1} times the Lebesgue measure of the set of such tuples. Fix x_0 and write $x_i = x_0 + y_i$, with y_1 the point of largest norm ζ (a factor d covers the choice). Then $\mathrm{vol} < \kappa(2\zeta)^d$, so y_d lies in a slab about the hyperplane through $0, y_1, \dots, y_{d-1}$ whose half-width is $\min(2\zeta, d\cdot\kappa(2\zeta)^d / V_{d-1}(0, y_1, \dots, y_{d-1}))$, where V_{d-1} is the $(d-1)$-volume, and every y_j lies in the ball $|y_j| \le \zeta = |y_1|$; the slab meets that ball in measure at most $C\kappa\zeta^{2d-1}/V_{d-1}$. Now V_{d-1} is $V_{d-2}(0, y_1, \dots, y_{d-2})$ times the distance from y_{d-1} to a flat of dimension $d-2$, over $d-1$. The reciprocal of the distance to a flat of codimension 2 is integrable over a ball in $\mathbb{R}^d$, and so on down to $V_1 = |y_1|$, so $\int dy_2\dots dy_{d-1}/V_{d-1} \le C\zeta^{(d-2)d-(d-1)}$. Integrating y_1 over $|y_1| \le 6\Lambda s$ and x_0 over the region gives $C\kappa$, with C depending on s. Markov's inequality gives the bound.

(a), (R3). By the Mecke equation, the expected number of pairs (σ, q) with σ as in (R2), $r_\sigma \le 3\Lambda s$, and q in the shell $\|q - c_\sigma| - r_\sigma| \le \gamma r_\sigma$, is at most Λ_0 times the shell's volume, at most $C\gamma\, s^d$, times the expected number of such σ, which is finite. ▮ for (a).

(b), step 1: quality. Let $\sigma = (p_0, \dots, p_d)$ have $\nu(\sigma) \ge \kappa$, diameter D_σ and radius r. Let $u_i = (p_i - c_\sigma)/r$ and U_σ the matrix with columns $u_i - u_0$. Points of a sphere of radius r within D_σ of one another lie in a cap of height at most $D_\sigma^2/(2r)$, so $\mathrm{vol}(\sigma) \le C\, D_\sigma^{d+1}/r$, hence $D_\sigma/r \ge \kappa/C$. Then $|\det U_\sigma| = d!\cdot\mathrm{vol}(\sigma)/r^d \ge c\kappa^{d+1}$, and since every column has length at most 2, the least singular value of U_σ is at least $\kappa' := c_d\kappa^{d+1}$.

(b), step 2: each Voronoi vertex moves a little. Let σ be a Delaunay simplex of P near U (its open ball is empty), with centre v and radius $r \le s/2$ (by (R1) every point near U has a keeper within $s/(4\Lambda)$). Put $F(w) = (D(w, p_i) - D(w, p_0))_{i=1\dots d}$. Then $|F(v)| \le 2\sqrt{d}\cdot\tau r(1+\tau)$, and the derivative of F at v for the flat distance is $-U_\sigma^{\mathsf T}$, of least singular value at least κ'. For $|w - v| \le r^* \le r/2$ the derivative of F differs from $-U_\sigma^{\mathsf T}$ by at most $2\sqrt{d}(\tau + 4r^*/r)$, since the second derivatives of D are at most $2/|w - p|$ there. With $r^* = 4|F(v)|/\kappa' \le 8\sqrt{d}\,\tau r/\kappa'$, this is at most $\kappa'/2$ once $\tau \le c_1\kappa'^2$, so $w \mapsto w + (U_\sigma^{\mathsf T})^{-1}F(w)$ is a contraction of factor ½ mapping the ball $B(v, r^*)$ into itself (Ortega and Rheinboldt 1970), and F has a unique zero w^* there. At w^* the $d+1$ distances $D(w^*, p_i)$ share one value R^* with $|R^* - r| \le C_3\tau r/\kappa'$. For q not in σ, (R3) and the empty ball give $|v - q| > r(1+\gamma)$, so $D(w^*, q) \ge (1-\tau)(r(1+\gamma) - C_3\tau r/\kappa') > R^*$ once $\gamma \ge C_2\tau/\kappa'$; points farther than $3\Lambda s$ have $D \ge 3s > R^*$. So w^* is a vertex of the cells of D, with sites exactly σ.

(b), step 3: and nothing else is a vertex. Let w near U be a vertex of the cells of D: $D(w, p_i) = R$ for $\sigma = (p_0, \dots, p_d)$ and $D(w, q) > R$ for every other q. By (R1), $R \le s/2$, so σ has diameter at most $6\Lambda s$ and $\nu(\sigma) \ge \kappa$ by (R2). Step 2 with the roles of D and the flat distance exchanged (the hypotheses are symmetric up to constants) gives a point v within $C_3\tau R/\kappa'$ of w equidistant from σ in the flat distance, at distance r. If some q had $|v - q| < r$, then $|w - q| < r + C_3\tau R/\kappa'$, while $D(w, q) > R$ gives $|w - q| > R/(1+\tau)$; so q would lie within $C_4\tau r/\kappa' < \gamma r$ of the sphere of σ, against (R3). So σ is a Delaunay simplex and w is the vertex step 2 gives it. Vertices of D near U and Delaunay simplices near U correspond one to one, with the same sites.

(b), step 4: neighbours are read from vertices. First, keepers near U are not too close: for keepers a and b whose cells meet U, so within $s/2$ of U, with $|a - b| \le s$ (otherwise there is nothing to prove),

(R1) gives d − 1 further keepers at distance about s/2 from a in independent directions, which with a and b form a tuple of diameter at most 2s and normalised volume at most C|a − b|/s, so (R2) gives |a − b| ≥ cκs. At a vertex of step 2 the d differences of distances have independent gradients, so near it the set where p_i and p_j tie and all others are farther is a piece of dimension d − 1 (implicit function theorem): every two sites of a vertex are neighbours. Conversely let keepers a and b, with cells meeting U, share a piece Φ of dimension d − 1. Its closure is compact, since cells near U have diameter at most s. Let f = D(·, a) on the set where D(·, a) = D(·, b). For the flat distance this set is the bisector and f² = |a − b|²/4 + |z − m|², with m the midpoint, whose second derivative along the set is at least |a − b|²/(4f³) ≥ cκ²/f, since f ≤ s/2. More generally, on the set where k + 1 keepers $p_0 = a$, $p_1 = b$, ..., p_k tie (1 ≤ k ≤ d − 1), the flat f² is the square of their circumradius r_k plus the squared distance to their circumcentre, so f's second derivative along the set is at least r_k²/f³ ≥ cκ²/f, since r_k ≥ |a − b|/2 ≥ cκs. The set is cut out by the k functions D(·, p_i) − D(·, p_0), whose flat gradients on it are the rows (p_0 − p_i)/f. Completed by d − k keepers from (R1) to a tuple of diameter at most 2s, the p_i are a face of a simplex with ν ≥ κ, by (R2); a k-face of a d-simplex has k-volume at least the simplex's volume over diam^(d−k), so the face's own normalised volume is at least κ, and the least singular value of the rows p_i − p_0 is at least c times that normalised volume times the face's diameter, so at least cκ|a − b|. Hence the rows (p_0 − p_i)/f have least singular value σ ≥ cκ², since |a − b| ≥ cκs and f ≤ s/2. A τ-close D moves the set, its tangent spaces and f's second derivatives along it by at most Cτ/(σ²f) ≤ Cτ/(κ⁴f) (implicit function theorem with these rows), so for τ ≤ cκ⁶, which the hypothesis gives since 2(d + 1) ≥ 6, f has no interior local maximum on any of these sets. So f attains its maximum over the closure of Φ on its relative boundary, where one more keeper ties, and so on at every step, down to a vertex that a and b share. So a and b are neighbours if and only if they share a vertex, for D, for D′ and for the flat distance alike, and step 3 makes the vertices, with their sites, the same in all three. ∎

Lemma 2.1 (the metric's distance is close to the frozen one). *In a ball B of radius r about x_0 on which |h(·, t) − h_0| ≤ η|h_0| for every t in a window, and in coordinates with h_0 = I, the distance d_t of Definition 1 is τ-close at scale sε, for every t in the window, with τ = C(η + sε).*

Proof. For points within 3Λsε of each other, the shortest path of h(·, t) between them stays within that distance, where h(·, t) differs from h_0 by at most η|h_0| and its first derivatives are bounded. So d_t(z, p) = |z − p|_h(p,t) + O(|z − p|²), and the distance and its first two derivatives in z differ from those of |z − p| by at most C(η + sε) times |z − p|, 1 and 1/|z − p| (first and second variation of length; e.g. Boissonnat, Dyer and Ghosh 2018, Section 3). The global lower bound D ≥ |z − p|/Λ is the eigenvalue bound on h. ∎

What it means. In a patch where the metric is within η of h_0, and at every moment of a window, a keeper's neighbours are those of the frozen world unless a nearly flat simplex or a nearly balanced set of d + 2 keepers sits nearby, which has probability of order a power of η.

3. Lemma G: Rare Costs along Any Walk

A walk may choose where to go, so it could seek out the places where detours cost most. Lemma G bounds the worst case over every connected set of cubes at once, when costly places are rare and do not depend on one another at a distance. It is a greedy-lattice-animal estimate in the spirit of Cox, Gandolfi, Griffin and Kesten (1993) and Martin (2002), proved directly in the form needed here.

Lemma G. *Index the cubes by ℤ^d. Let X_Q ≥ 0, where X_Q is a function of independent random elements attached to the cubes within index distance K of Q. Suppose P(X_Q > 0) ≤ q and E[e^(2X_Q)] ≤ M for every Q. Then there are C and C_0, depending only on d and K, such that for every y ≥ C M^(1/2), with probability at least 1 − C exp(−y/C), every ∗-connected set A of cubes that contains the cube of the origin has Σ_(Q ∈ A) X_Q ≤ C_0M^(1/2)|A| q^(1/(2d)) + y.*

Proof. Let L = ⌈q^(−1/(2d))⌉ and group the cubes into blocks of L^d. Step 1: A meets a ∗-connected set of at most C_d(1 + |A|/L) blocks. Take a maximal set of cubes of A pairwise at index distance at least L. Balls of radius L/2 about them are disjoint, and each holds at least L/2 cubes of A, since A is

connected and reaches beyond it, unless A lies inside one such ball. So there are at most max(1, 2|A|/L) of them, every cube of A lies within L of one, and each accounts for at most C_d blocks. Step 2: fix a ∗-connected set $\mathfrak{B}$ of B blocks and let S be the sum of X_Q over its cubes. Split the cubes into (2K + 1)^d residue classes; within a class the X_Q are independent. E[e^(X_Q)] ≤ 1 + P(X_Q > 0)^(1/2)E[e^(2X_Q)]^(1/2) ≤ 1 + q^(1/2)M^(1/2), so a class's part S_c of S, a sum over at most BL^d cubes, has E[e^(S_c)] ≤ exp(BL^d q^(1/2) M^(1/2)) ≤ exp(2^d B M^(1/2)). With x_0 = 2^d M^(1/2) + log C_1 + 2, Markov's inequality gives P(S_c > x_0B + y′) ≤ exp(B(2^d M^(1/2) − x_0) − y′) = exp(−B(log C_1 + 2) − y′). There are at most C_1^B such $\mathfrak{B}$ containing a given block, so the chance that some $\mathfrak{B}$ containing the origin's block has a class with S_c > x_0B + y′ is at most (2K + 1)^d Σ_B e^(−2B − y′) ≤ C e^(−y′). Off that event, for every A, Σ_A X_Q ≤ S($\mathfrak{B}$(A)) ≤ (2K + 1)^d (x_0B + y′), with B ≤ C_d(1 + |A|/L) ≤ C_d(1 + 2|A| q^(1/(2d))). Since M ≥ 1, x_0 ≤ CM^(1/2), so this is at most C_0M^(1/2)|A| q^(1/(2d)) + y with y = C(M^(1/2) + y′). ∎

4. Lemmas U and D: Comparing a Patch with Its Frozen World

Lemma U (uniform constants). *Let W be the world of Definition 1 at a fixed time, or a graph on its keepers that contains the neighbours at some time t_1 of a window of length at most 1 and every edge of which joins keepers that are neighbours at some time of that window. Then, on every compact set and with constants depending only on d, λ±, N± and the window, W has the conclusions of FR16 Lemmas A.0 (for the world at a fixed time), A.1, A.2, A.4 and A.6, and the static form of Lemma A.7: for every compact K and δ > 0, the chance that some keepers j and k in K within δ of each other have ρ(p_j, p_k) > C(δ + ε) tends to 0 as ε → 0.*

Proof. FR16's proofs of these lemmas use the world's law only through W2, W3 and Lemma A.0, and use W1 only to say that certain counts have one law at every point, where a uniform bound serves equally. A.0: cells of one metric are star-shaped about their keepers and meet along smooth hypersurfaces; a generic segment meets the pieces of dimension d − 2 or less of finitely many cells in a set of measure zero, and crosses from one cell to the next through a point that only two cells hold, so the cells it meets form a walk. W2: a cell meeting a unit ball has its keeper within the largest distance from a point of the ball to its nearest keeper, which has the empty-ball tail of FR16 Lemma 5.3 with the intensity bounds and the metric's eigenvalue bounds, so cell counts have moments of every order, and an exponential moment, uniformly. W3: Proposition A.9's proof applies with its local event E_Q, read with the constants of (R1): off E_Q, every point near Q has a keeper within s/(4Λ), so by star-shapedness every cell meeting Q has diameter at most s, and E_Q depends only on the keepers within 2s of Q, so cubes far apart are independent. For a graph containing the neighbours at t_1, every walk at t_1 is a walk of W, so the upper bounds A.4, A.6 and A.7 carry over; for the lower bound A.2(a), each edge joins two cells that share a face at some time of the window, and A.2's chain of cubes is built from a point of that shared face, with "bad" read over the window, which keeps W3's bound with the window's constants. ∎

Lemma D (the comparison). *Fix a point x_0, a radius r, the frozen world F at x_0 (Definition 3), and a world V on the same probability space with a fixed neighbour graph, such that keepers present in both worlds have V-waits between 1 − η and 1 + η times their F-waits. Let $\mathcal{A}$, the affected keepers, be any set containing every keeper not present in both worlds and such that two keepers outside $\mathcal{A}$ are neighbours in V exactly when they are neighbours in F. Let $\hat{Z}_Q$ be indicators of cubes of side sε, each a function of the randomness within Ksε of Q, with P($\hat{Z}_Q$ = 1) ≤ q, and equal to 1 whenever Q is bad for F in the sense of W3 (in (ii), for V too) or the cell of some affected keeper meets Q; outside B(x_0, 2r) let $\hat{Z}_Q$ be independent indicators of probability q. Fix u and v in B(x_0, r/C_0). Then, for every y ≥ C(s):*

(i) ρ_V(u, v) ≤ (1 + η)ρ_F(u, v) + C(s)q^(1/(2d))|u − v| + C(s)yε, outside an event of probability at most C exp(−y/C) + o(1) as ε → 0 and off the endpoint events of u and v. The endpoint event of u is that Q(u) lies within distance 2 of a $\hat{Z}$-cluster or in a bounded region that one encloses; it has

probability at most Cq. When V's neighbours are the cells of a tiling, as for the world of Definition 1 at a fixed time, the endpoint events can be dropped;

(ii) if V has the conclusions of Lemma U, the same holds with V and F exchanged, without the term Cq.

In every use below y = ε^(−1/2), so C(s)yε and exp(−y/C) are o(1) as ε → 0, and Lemma D's failure probability is o(1), apart from the endpoint events in (i) for a V that is not a tiling.

Proof. (i) Let γ be F's cheapest walk from k(u) to k(v), with n keepers c_0, …, c_n. By FR16 Lemmas A.6 and A.2 (Count 2), outside an event of probability o(1), ρ_F(u, v) ≤ C|u − v| + Cε and n ≤ C|u − v|/ε + C. Build γ's chain of cubes as in FR16 Lemma A.2: points y_i in the shared face of c_(i−1) and c_i, consecutive cubes Q(y_i) equal or ∗-neighbours when the cell between them is small, and a ∗-path of bad cubes through it when it is wide. Apply the splice of FR16 Lemma 5.8 (hulls and Timár's theorem), with "not fine" read as $\check{Z} = 1$, to this chain: the result is a chain from Q(u) to Q(v) whose cubes all have $\check{Z} = 0$, made of cubes of the original chain outside the spliced hulls and of cubes on the outer boundaries of $\check{Z}$-clusters, each touching its cluster, joined through shared faces. If Q(u) or Q(v) lies within distance 2 of a $\check{Z}$-cluster, or in a bounded region that one encloses, the chain starts or ends instead at the first clean cube outside that cluster's hull, reached from u (or v) by the cells met by a segment. Those cells form a walk of whichever world is a tiling, the hull at Q(u) has more than m cubes only with probability at most (Cq)^(m/C), and so this piece costs o(1) as ε → 0. In (i) for a V that is not a tiling no such walk need exist, so (i) is stated off the endpoint events, each of probability at most Cq. By A.6 and A.2, γ stays within C|u − v| ≤ r/2 of u, outside probability o(1), and a $\check{Z}$-cluster meeting the chain and reaching beyond $B(x_0, 3r/2)$ needs a run of order r/(sε) cubes with $\check{Z} = 1$, of probability o(1); so only the patch's own indicators matter.

Turn the chain into a walk of V. Along a stretch of original cubes kept, use γ's own keepers: each has a cell meeting a cube with $\check{Z} = 0$, so it is unaffected, and consecutive unaffected keepers of γ are neighbours in F, hence in V. Where the chain leaves γ at a point y_i, follows a cluster's boundary, and rejoins γ at y_j, use the cells met by the segments from y_i to the centre of its cube, between the centres of consecutive boundary cubes, and from the last centre to y_j. Each segment lies in cubes with $\check{Z} = 0$, so the cells it meets are unaffected; they form a walk of F by FR16 Lemma A.0, hence of V. The walk's cost is at most (1 + η)ρ_F(u, v) plus the waits of the cells meeting the cubes within distance 3 of each spliced cluster. Those cubes have $\check{Z} = 0$ and are good, so their cells have keepers within 4sε of a cube of the cluster. The extra cost is therefore at most (ε/N-) Σ_(Q ∈ A) $\check{Z}$_Q W_Q, where A is the union of the original chain's cubes Q(y_i) with every $\check{Z}$-cluster spliced, a ∗-connected set containing Q(u), and W_Q is the total of the Exp(1) clock variables of the keepers of either world within 5sε of Q, a function of the randomness near Q.

Apply Lemma G twice. First with X_Q = $\check{Z}$_Q: Σ_A $\check{Z}$_Q ≤ C_0|A|q^(1/(2d)) + y, and since every cube of A outside the Q(y_i) has $\check{Z} = 1$ (the bad cubes that bridge a wide cell included, since $\check{Z} = 1$ on bad cubes, so they lie in the spliced clusters), |A| ≤ 2n + 2 + C_0|A|q^(1/(2d)) + y, so |A| ≤ 2(2n + 2 + y) once $C_0 q^{1/(2d)} \le \tfrac{1}{2}$. Second with X_Q = t(s)$\check{Z}$_Q W_Q. W_Q is a sum of N_Q independent Exp(1) variables, where N_Q, the number of keepers within 5sε of Q, is Poisson with bounded mean, so E[e^(2t(s)W_Q)] ≤ M(s) for t(s) > 0 small enough; and since $\check{Z}$_Q is 0 or 1 and W_Q ≥ 0, E[e^(2t(s)$\check{Z}$_Q W_Q)] ≤ E[e^(2t(s)W_Q)] ≤ M(s), whatever the dependence between them (crowded cubes have both $\check{Z} = 1$ and large W). The second sum is then at most (C_0M(s)^(1/2)|A|q^(1/(2d)) + y)/t(s). With n ≤ C|u − v|/ε + C, the extra cost is at most C(s)q^(1/(2d))|u − v| + C(s)yε. The failure probabilities add to the stated bound.

(ii) The same construction from V's cheapest walk, with FR16's lemmas for V (the chain of cubes, A.2 and A.6) supplied by Lemma U. Stretches of V's walk through cubes with $\check{Z} = 0$ use unaffected keepers, whose V-edges are F-edges; the detours are built from F's cells exactly as in (i). The result is a walk of F. ∎

What it means. Two worlds that agree except on a sparse set of affected keepers have the same passage, up to a cost proportional to the distance and to a positive power of the density of trouble. The cheapest walk may go anywhere, and Lemma G covers every route at once, so nothing depends on how the walk was chosen.

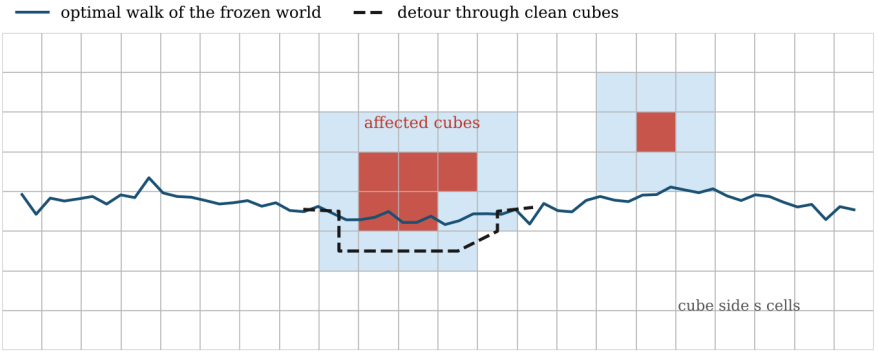


Figure 1. The detour of Lemma D. The blue line is the cheapest walk in the frozen world. Red cubes hold an affected keeper; the pale ring around each red cluster is clean. In clean cubes the real world and the frozen world have exactly the same neighbours, so the walk is kept as it is there and goes round each red cluster through its ring. Lemma G bounds the total cost of every ring a walk can meet.

5. Theorem B: The Static Bridge

Theorem B. *Let h and N not depend on t. (i) For fixed $x \neq y$, $\rho_\varepsilon(x, y) \to d(x, y)$ in probability as $\varepsilon \to 0$, and the same holds after finitely many keepers are added at fixed points. (ii) For every compact K, sup over x, y in K of $|\rho_\varepsilon(x, y) - d(x, y)| \to 0$ in probability. (iii) Let V be a world on the same keepers, up to keepers of density $o(\varepsilon^{-d})$, that satisfies Lemma D's hypotheses with respect to the frozen worlds of every patch with $q \le q_0 + o(1)$ as the patch shrinks and $\varepsilon \to 0$. If V's neighbours are the cells of a tiling, the upper bound in (i) holds for V up to an error $\omega(q_0)|x - y|$. If not, it holds at the scale of one patch: for fixed u and v in $B(x_0, r/C_0)$, outside an event of probability $o(1)$ as $\varepsilon \to 0$ and off Lemma D's endpoint events of u and v, $\rho_V(u, v) \le |u - v|_{h(x_0)} / (c_0 N(x_0)) + (C\eta(r) + \omega(q_0))|u - v| + o(1)$. If V also has Lemma U's conclusions, the lower bounds in (i) and (ii) hold for V up to the error $\omega(q_0)|x - y|$. Here $\omega(q_0) \to 0$ as $q_0 \to 0$.*

Proof. Step 1 (a patch). Fix x_0 and r, and let $\eta(r)$ be the largest relative change of h and N over $B(x_0, 2r)$. Couple the world and the frozen world F at x_0 by building both from one Poisson process of intensity $\varepsilon^{-d} \max \sqrt{\det h}$ on $B(x_0, 2r)$ with independent uniform marks (a point belongs to a world when its mark is below that world's intensity over the maximum) and one family of Exp(1) clock variables. Keepers present in only one world have intensity at most $C\eta(r)\varepsilon^{-d}$. Let $\tau = C(\eta(r) + s\varepsilon)$ as in Lemma 2.1, $\kappa = \tau^{1/(3(d+1))}$ and $\gamma = \tau^{1/3}$, which meet Lemma M(b)'s conditions for small τ. Let Z'_Q be 1 when Q's local event E_Q of Lemma U holds for either world, or a keeper within $6\Lambda s\varepsilon$ of Q is present in only one world, or Q is not (κ, γ)-regular at scale s for the world's keepers. Call affected every keeper present in only one world and every keeper whose cell, in either world, meets a cube with $Z' = 1$, and let $\tilde{Z}_Q$ be the largest Z' over the cubes within distance 1 of Q. If $\tilde{Z}_Q = 0$, Q is good, so the cells meeting it are small and meet only cubes with $Z' = 0$: they are unaffected, as Lemma D asks. Two unaffected keepers have the same neighbours in both worlds: near them the two worlds have the same keepers and the configuration is regular, so by Lemma 2.1 and Lemma M(b) the cells of the world's distance, of the h_0-distance, and of F are combinatorially the same there. Waits differ by at most a factor $1 \pm \eta(r)$. $\tilde{Z}_Q$ is a function of the randomness within $Ks\varepsilon$ of Q, and by Lemma M(a), $P(\tilde{Z}_Q = 1) \le q(r, s) := C \exp(-cs^d) + C(s)(\eta(r) + s\varepsilon)^{1/(3(d+1))}$. Lemma D applies with $y = \varepsilon^{-1/2}$, both worlds being tilings, and with F's limit from Lemma 1.1: for fixed u and v in $B(x_0, r/C_0)$, outside an event of probability $o(1)$ as $\varepsilon \to 0$,

$|\rho_\varepsilon(u, v) - |u - v|_h(x_0)/(c_0 N(x_0))| \le (C\,\eta(r) + C(s)\,q(r, s)^{(1/(2d))})|u - v| + o(1).$

Lemma M(a)'s constant C(s) grows only as a power of s (it counts tuples in a ball of radius of order s), so s = s(r) can be chosen to tend to infinity as r → 0 slowly enough that $C(s)q(r, s)^{(1/(2d))} \to 0$, for instance s of order $(\ln(1/\eta(r)))^{(1/(2d))}$. Write ω(r) for the resulting error factor; it tends to 0 as r → 0, and the failure probability π(r) is o(1) as ε → 0.

Step 2 (upper bound for fixed points). Take a polygonal path from x to y whose optical length is within r of d(x, y), with vertices $z_0 = x, \ldots, z_J = y$, consecutive vertices within r/C_0. By the triangle inequality $\rho_\varepsilon(x, y) \le \sum_j \rho_\varepsilon(z_j, z_{j+1})$, and by Step 1 at each $x_0 = z_j$, outside probability Jπ(r) + o(1), the sum is at most $\sum_j |z_{j+1} - z_j|_h(z_j)/(c_0 N(z_j)) + \omega(r)\cdot\text{length} + o(1) \le d(x, y) + \omega'(r) + o(1)$, by continuity of h and N. J does not depend on ε. Let ε → 0, then r → 0. Keepers added at fixed points are present in only one world, so they enter $\hat{Z}$ at boundedly many cubes and change nothing in the limit.

Step 3 (lower bound, uniformly). Let δ > 0 and let $\mathbb{G}$ be a grid of spacing δr. By Lemma U and FR16 Lemma A.2, outside probability o(1), for all x and y in K the cheapest walk from k(x) to k(y) stays in a compact K′, and the static A.7 gives $\rho_\varepsilon(p_j, g) \le C(\delta r + \varepsilon)$ for every keeper j in K′ and every grid point g within δr of it. Mark the walk at keepers $m_0 = k(x), m_1, \ldots$: m_{i+1} is the first keeper after m_i farther than $r/(2C_0)$ from it, and the last mark is k(y). By A.2(a), applied from every keeper of K′ at once (its failure probability is exponentially small in r/ε), every stretch between marks costs at least $a\cdot r/(2C_0)$, so there are at most $J \le C\rho_\varepsilon(x, y)/r + 1 \le C/r$ marks. Cutting the walk at the marks double-counts their waits, each at most Cε log(1/ε) outside probability o(1). With g_i the grid point nearest m_i, the triangle inequality gives $\rho_\varepsilon(x, y) \ge \sum_i \rho_\varepsilon(g_i, g_{i+1}) - JC(\delta r + \varepsilon \log(1/\varepsilon))$. For $\delta \le 1/(4C_0)$ the pairs (g_i, g_{i+1}) range over a finite set of grid pairs within r/C_0 of each other, fixed as ε → 0, and Step 1 with Lemma D(ii) bounds each from below at once: $\rho_\varepsilon(g, g') \ge |g - g'|_h(g)/(c_0 N(g)) - \omega(r)|g - g'| - o(1)$. The grid polygon runs from within δr of x to within δr of y, so its optical length is at least $d(x, y) - C\delta r$. Altogether $\rho_\varepsilon(x, y) \ge d(x, y) - C(\delta + \omega(r)) - o(1)$ for all x and y in K at once. The upper bound holds uniformly by the same grid: $\rho_\varepsilon(x, y) \le \rho_\varepsilon(g_x, g_y) + 2C(\delta r + \varepsilon)$, with Step 2 for the finitely many grid pairs. Let ε → 0, then r → 0, then δ → 0.

Step 4 (other worlds). Steps 1 to 3 use of the world only Lemma D's hypotheses and, for the lower bounds, Lemma U's conclusions. With $q \le q_0 + o(1)$, Step 1's error factor is $C(s)(q_0 + o(1))^{(1/(2d))} + C\eta(r)$, where q_0 itself contains the term $C\exp(-cs^d)$; choosing s as a function of q_0's other part, tending to infinity slowly as that part tends to 0, makes the factor at most $\omega(q_0) \to 0$. Lemma D's failure probability is o(1) in direction (ii), and in direction (i) for tilings; for a V that is not a tiling, Lemma D(i) holds off its endpoint events, and Step 1 alone gives the patch-scale bound. Steps 2 and 3 carry the rest through, which gives (iii). Step 2's polygon is not used for such a V: each of its vertices would need an endpoint condition, and their number grows as r → 0. ∎

What it means. Static held cells whose sizes, shapes and clock rates vary slowly make, at large scale, the static spacetime those sizes and rates describe, round cells and shaped cells alike. Lemma D compares whole worlds and Lemma G covers every walk at once, so no defect has to be treated one at a time.

6. Theorem W: Shape That Changes in Time

Theorem W. *In the world of Definition 1, for a fixed source point (x_0, t_0) with source episode x, the first tick of $k(x_0)$ after t_0, and a fixed target point y, the first tick of k(y) in the future of x comes at $T^{g(y)} + o(1)$ in probability as ε → 0. The same holds for the past of an episode, with the receiver's convention, and after finitely many keepers and ticks are added at fixed points.*

Slabs. Fix Δ > 0 and cut time into slabs $[t_i, t_{i+1})$, $t_i = t_0 + i\Delta$. For a window I, let $G_\cup(I)$ be the graph of all pairs that are neighbours at some time of I, and $G_\cap(I)$ the graph of the pairs that are neighbours throughout I. A push at a time s of I runs along an edge present at s, so every chain of the

world is a chain of the world whose graph is G_∪ of the slab holding each push, and every chain of a world whose graph over I is G_∩(I) is a chain of the world.

Lemma 6.1 (a slab). *In a world whose graph is a fixed G on [t_i, t_(i+1)], let R_i be the keepers with a tick in the future of x by time t_i. Every keeper b first reached in the slab is reached at t_i + ρ_i(R_i, b) := t_i + min over a in R_i of the least total of waits over walks of G from a to b, where a keeper of R_i waits from t_i to its next tick and every other keeper from the push that activates it to its next tick. Given everything up to t_i, these waits are independent, and each lies between E_k ε/((1 + CΔ)N(x_k, t_i)) and E_k ε/((1 − CΔ)N(x_k, t_i)), with E_k independent Exp(1) variables independent of the past.*

Proof. This is FR16 Lemma 5.2's first-passage recursion started from the set R_i, all of whose keepers are active at t_i. Its proof, by memorylessness of Poisson clocks, applies unchanged with rates that vary in time: a clock of rate ε^(−1)N(x_k, t) is a rate-1 clock in the time Λ_k(t) = ε^(−1)∫N(x_k, s) ds, so the wait from any stopping time to the next tick is Λ_k^(−1)(Λ_k(·) + E_k) minus that time, and over a slab N varies by at most a factor 1 ± CΔ. ∎

Proof of Theorem W. **Lower bound.** Since every chain of the world is a chain of the slab-union world, the world's arrival times are at least that world's, T ≥ T_∪. In slab i the slab-union world's graph is G_∪([t_i, t_(i+1)]); call its passage ρ_i, computed with the waits E_kε/N(x_k, t_i) of Lemma 6.1. G_∪ has Lemma U's conclusions (it contains the neighbours at t_i, and every edge joins keepers that are neighbours during the slab). It satisfies Lemma D's hypotheses with respect to the frozen worlds at (x_0', t_i), with Lemma 2.1 read over the slab: η is the change of h and N over the patch and over the slab, at most C(r + Δ), and by Lemma M(b) a regular region's neighbours are the frozen world's at every moment of the slab. Take patches of radius r = Δ and a grid of spacing δΔ, and choose the cube size s = s(Δ) as in Theorem B's Step 4. Theorem B(iii) applies, in its lower-bound direction (which builds walks of the frozen world and so needs no condition at the endpoints), with the static metric h(·, t_i), rate N(·, t_i) and q_0 = C exp(−cs^d) + C(s)Δ^(1/(3(d+1)))): uniformly over keepers a, b in a compact set, outside probability o(1) as ε → 0, ρ_i(a, b) ≥ d_i(a, b) − (ω_Δ + Cδ)|a − b| − Cδ Δ − o(1). Here d_i is the optical distance of h(·, t_i) and N(·, t_i), ω_Δ → 0 as Δ → 0, and the δ terms are Theorem B's Step 3 grid errors, which with patches of radius Δ are proportional to the distance plus at most CδΔ. Write E_i for the larger of 0 and the largest of T^g(p_b) − T_∪(b) over keepers b reached by t_i in a fixed compact set (the top speed, Lemma U, keeps the reached keepers in one). Let b be first reached in slab i. By Lemma 6.1 and the sandwich of its waits, T_∪(b) ≥ t_i + (1 − CΔ)ρ_i(a, b) for some keeper a reached by t_i, and ρ_i(a, b) ≤ CΔ since b is reached within the slab. Staying at p_a is causal for g, so T^g(p_b) ≤ max(T^g(p_a), t_i) + d_i(a, b)(1 + CΔ), using that over the times [t_i, t_i + CΔ] a g-null curve's speed differs from that of the metric frozen at t_i by a factor 1 ± CΔ. With T^g(p_a) ≤ T_∪(a) + E_i ≤ t_i + E_i, this gives T^g(p_b) − T_∪(b) ≤ E_i + CΔ(Δ + ω_Δ + δ) + o(1). So E_(i+1) ≤ E_i + CΔ(Δ + ω_Δ + δ) + o(1), and over the at most C/Δ slabs up to the target's time E ≤ C(Δ + ω_Δ + δ) + o(1), the number of slabs being fixed as ε → 0. Hence T(k(y)) ≥ T^g(y) − C(Δ + ω_Δ + δ) − o(1). Let ε → 0, then δ → 0, then Δ → 0.

Upper bound. The always-present graphs G_∩ are not tilings: a keeper whose neighbours all change during a window has no edge in them. So their legs need endpoints off Lemma D's endpoint events, and Theorem B(iii) gives them the upper bound only at the scale of one patch. Points. Choose points z_0 = x_0, z_1, …, z_J = y on a g-causal curve that realises T^g(y): z_1 and z_(J−1) at T^g = t_0 + δΔ and T^g(y) − δΔ, and between them T^g(z_(j+1)) − T^g(z_j) between Δ/2 and 2Δ, so that J ≤ C/Δ, fixed as ε → 0. Each leg then spans a distance at most C_2Δ, and the two short legs at most C_2δΔ. For each z_j with 0 < j < J, fix in advance M points within δ²Δ of z_j, pairwise more than δ³Δ apart; z′_j will be one of them, and z'_0 = x_0, z′_J = y. Start times and windows. Put S_0 = t_0 and S_(j+1) = S_j + T^g(z_(j+1)) − T^g(z_j) + C_3Δ_j(Δ_j + δ + ω_j + S_j − T^g(z_j)), with Δ_j = Δ and ω_j = ω_Δ, except Δ_j = δΔ and ω_j = ω_(δΔ) for the two short legs; these are numbers fixed in advance. Leg j uses the fixed graph G_j = G_∩([S_j, S_j + 3Δ_j]), the frozen world at (z_j, S_j) and a patch of radius C_0C_2Δ_j about z_j, so that every candidate pair of the leg lies in B(z_j, C_2Δ_j); G_j satisfies Lemma D's hypotheses there with η ≤ CΔ_j, since a regular region's neighbours are the frozen world's throughout the window.

Write $\tilde{Z}^{(j)}$ for that leg's indicators, with $q \le q_0 + o(1)$, and $q(\delta\Delta)$ for q_0 with patches and windows of size $\delta\Delta$, which tends to 0 as $\delta \to 0$. Clean endpoints. Let $R = C_4 \log(1/\varepsilon)$. Call a candidate w good when, for each of the two legs that meet it, no cube within distance 2 of Q(w) has $\tilde{Z} = 1$ and Q(w) is not cut off from the boundary of the box of 4R cubes about it by cubes of that box with $\tilde{Z} = 1$. Goodness is decided by the randomness within $(2R + K)s\varepsilon$ of w, which tends to 0, so for ε small the M candidates near z_j are good independently, each with probability at least $1 - C(q_0 + q(\delta\Delta))$ by the Peierls bound in Lemma D's proof. Let $\mathcal{G}$ be the event that, for every leg, no $\tilde{Z}$-cluster of diameter at least R cubes meets the compact set; by the same bound $P(\text{not } \mathcal{G}) \le CJ\varepsilon^{(-d)}(Cq_0)^{(R/C)}$, which tends to 0 for C_4 large, q_0 being fixed as $\varepsilon \to 0$. On $\mathcal{G}$ a good candidate is off the endpoint events of both its legs, since a cluster that encloses Q(w) and has diameter less than R lies in the box. Let z'_j be the first good candidate near z_j. All M fail for some j with probability at most $J(C(q_0 + q(\delta\Delta)))^M$, which is at most δ for M fixed and large enough. The fixed points x_0 and y cannot be chosen: they are off the endpoint events of the short legs outside probability $Cq(\delta\Delta)$. The legs. Theorem B(iii), at the scale of one patch, applies at once to the at most M^2 fixed pairs of candidates of each leg (M for the short legs), by a finite union bound, outside probability o(1) as $\varepsilon \to 0$; so it holds for the pair (z'_j, z'_{j+1}) actually chosen, which is off its endpoint events. Suppose $k(z'_j)$ has a tick in the future of x by S_j, or $j = 0$. Its first tick after S_j is then in the future of x, since its ticks are carried, or is x itself, and from it pushes run along G_j, so $k(z'_{j+1})$ is reached by $S_j + \rho_{G_j}(k(z'_j), k(z'_{j+1}))$, computed with the waits after S_j, provided this is inside the window. S_j is a fixed time, so given the keepers these waits are fresh, and over the window the rates vary by a factor $1 \pm C\Delta_j$. Outside probability o(1), the leg takes at most $(1 + C\Delta_j)|z'_{j+1} - z'_j|_{h(z_j, S_j)} / (c_0 N(z_j, S_j)) + (\omega_j + C\Delta_j)C_2\Delta_j + o(1)$. The curve's piece from z_j to z_{j+1} is g-causal over the time $T^g(z_{j+1}) - T^g(z_j) \le 2\Delta_j$ from $T^g(z_j)$, so its optical length in the metric frozen at (z_j, S_j) is at most that time multiplied by $1 + C(\Delta_j + S_j - T^g(z_j))$, and the frozen distance between the candidates is at most that plus $C\delta^2\Delta$. For ε small the o(1) is below Δ_j^2, so the leg ends by S_{j+1} once C_3 is large, which is before $S_j + 3\Delta_j$. Induction from $j = 0$, where x is the first tick of $k(x_0)$ after $S_0 = t_0$, gives k(y) reached by S_J. With $\lambda_j = S_j - T^g(z_j)$, $\lambda_{j+1} = \lambda_j(1 + C_3\Delta_j) + C_3\Delta_j(\Delta_j + \delta + \omega_j)$, so by Grönwall's inequality $\lambda_J \le C(\Delta + \delta + \omega_\Delta)$, with C depending on $T^g(y) - t_0$. So the first tick of k(y) in the future of x comes by $T^g(y) + C(\Delta + \delta + \omega_\Delta)$, outside probability $o(1) + \delta + Cq(\delta\Delta)$. The start times are fixed for a reason: each leg's error is of order ω_Δ times its length, so the lag built up over the legs is of order ω_Δ, which need not be small against Δ; a window placed at $t_0 + j\Delta$ would not hold the later legs, and a window placed at S_j does. Let $\varepsilon \to 0$, then $\delta \to 0$, then $\Delta \to 0$.

Past, and added points. Reversing time turns the past of an episode into a future with the receiver's convention; the slab inclusions hold for it too (the inclusions hold under either convention), and the proof is the same. Added ticks change no cell. Counted as present in one world only, added keepers enter $\tilde{Z}$ at boundedly many cubes, which is harmless where Lemma D compares tilings. In the upper bound it is not: there the cube of an added keeper at x_0 or y would always be in trouble, so it could not be a clean endpoint. So there the added keepers are put into the frozen worlds as well. They are then not affected for being present in one world; the cube of an added keeper at x_0 or y is clean outside probability $Cq(\delta\Delta) + P(\text{not } \mathcal{G})$, by Lemma M(a) and Lemma U for a Poisson process with finitely many fixed points added, since the Mecke equation counts the tuples that contain a fixed point in the same way; and a frozen world with the added keepers has Lemma 1.1's limit, by Lemma D between it and the frozen world without them, both tilings. Theorem O uses this form, with the keeper added at the target (Step 2) or at the source (Step 4). ∎

What it means. Cells whose shape changes in time, with keepers fixed, clocks given and the size of space fixed, make at large scale the order of the spacetime that shape describes, every first-order gravitational wave included. In a regular region a cell's neighbours do not change at all during a short window (Lemma M), so no neighbour changes inside a window, and the slabs carry the rest.

7. Theorem O: The Order Reads g

Theorem O. *In the world of Definition 1, let D be the causal diamond of g from (x_0, t_0) to (x_1, t_1), with (x_1, t_1) in the chronological future of (x_0, t_0) and D compact. Let x be the first tick of $k(x_0)$ after t_0 and y the first tick of $k(x_1)$ after t_1. Let μ be the measure $\sqrt{\det h}\cdot N\,dx\,dt$, so that $\mu = \sqrt{-g}\,dx\,dt / c_0$. Then, in probability as $\varepsilon \to 0$:*

(1) ε^(d+1)|I(x, y)| → μ(D);

(2) the share of related pairs in I(x, y) tends to the μ⊗μ-share of pairs of D related in g;

(3) for every k ≥ 2, the number of k-chains in I(x, y), divided by the number of k-subsets, tends to the μ^k-share of k-tuples of D that form a chain of g in some order.

Proof. Step 1 (counting). For a bounded set B whose boundary has μ-measure zero, the number of episodes in B has mean ε^(−(d+1))μ(B), and, the keepers being Poisson and the clocks independent Poisson given the keepers, variance at most Cε^(−(d+2)). By Chebyshev's inequality, ε^(d+1) times the count tends to μ(B) in probability. The same holds for counts of k-tuples in products of such sets, which are products of counts up to diagonal terms of lower order.

Step 2 (few misjudged episodes from x). Call an episode w = (k, t) misjudged from x, with error ε′, when $x \prec w$ but $t < T^{g}(x_k) - \varepsilon'$, or $x \not\prec w$ but $t > T^{g}(x_k) + \varepsilon'$. Let Y_k be the distance between the time keeper k is first reached and T^g(x_k). If Y_k ≤ ε′, k has no misjudged episode, since every tick after the first one reached is reached by carries. So the number misjudged at k is at most the number n_k of k's ticks in the bounded time range of D times 1{Y_k > ε′}. By the Mecke equation for the Poisson keepers, the expected total over keepers in a bounded set is ∫ E[n_p 1{Y_p > ε′}] ε^(−d)√det h(p) dp, computed with a keeper added at p, and by Cauchy–Schwarz each integrand is at most Cε^(−1)P(Y_p > ε′)^(1/2). For each p ≠ x₀, P(Y_p > ε′) → 0 by Theorem W, with the keeper added at p. By dominated convergence the total is o(ε^(−(d+1))). This truncation needs convergence in probability only; no moment of the passage is used.

Step 3 (the past of y). The same, by Theorem W for the past.

Step 4 (few misjudged pairs). By the Mecke equation for the keepers and then for each clock given the keepers, the expected number of pairs (z, w) of episodes in a bounded set with w misjudged from z is the integral over source points (q, s) of the expected number misjudged from an episode added at time s to a keeper added at q, times the density of episodes. By Step 2's argument with that source, each integrand is o(ε^(−(d+1))) for fixed (q, s) and at most Cε^(−(d+1)), so by dominated convergence the total is o(ε^(−2(d+1))).

Step 5 (assembly). As in FR16 Theorem 5.10's proof: by Lemma U and FR16 Lemma A.2, every episode of I(x, y) lies in a fixed bounded region, outside probability o(1); x and y lie within ε′ of the tips, outside probability o(1). An episode of D farther than 2ε′ from its boundary follows x and precedes y unless it is misjudged, and one farther than 2ε′ outside D fails one of the two unless it is misjudged. The boundary of D is a union of two null hypersurfaces of g and has μ-measure zero, so the strip within 2ε′ of it holds a share of D that tends to 0 with ε′ (Step 1). This gives (1). Two episodes of I(x, y) are related as g says unless the pair is misjudged (Step 4) or lies within 2ε′ of g's light cone, a share that vanishes with ε′ (Step 1 for pairs). This gives (2). A k-tuple of I(x, y) forms a chain exactly when all its pairs are related consistently. The k-tuples containing a misjudged pair, or a pair near the cone, number at most the number of such pairs times |I(x, y)|^(k−2), which is o(ε^(−k(d+1))). The remaining k-tuples form a chain of the order exactly when they form a chain of g, and Step 1 counts them. This gives (3). Let ε → 0, then ε′ → 0. ∎

Corollary C1 (chain shares in every frame). *In FR16's flat world, for every k the number of k-chains in a large interval, divided by its Minkowski value, tends to 1 in probability in every frame.*

Proof. Theorem O with h = δ and N = 1, where g is Minkowski space with light speed c_0 and D is a Minkowski diamond in any frame; equally, FR16 Lemma 5.7(3) and Theorem 5.10's steps with Step 5's counting of k-tuples. ∎

What it means. The order of a world of Definition 1 reads, in every large interval, the count, the share of related pairs and the chain shares of g, whatever the interval's axis. The order of limits is ε → 0 first, the cell small against the interval, and then the interval small against the curvature.

8. Theorem LE: Density Follows Extent in the Tension World

Lemma 8.0 (scale covariance). *Let every length in a rule of held cells be a fixed multiple of the keepers' extent δ (FR15, H2; FR16 Definition 8), and let every act happen at a keeper's own episodes. Then the world with extent δ and pace p is the world with extent 1 and pace 1 under $x \mapsto \delta x$ and $t \mapsto t/p$, and its density is $n_1\delta^{(-d)}$ exactly, whatever p is.*

Proof. Reach is counted in crossings and its rest values in crossings (FR16 Definitions 7 and 9), so neither changes under x ↦ δx. The least separation and the cut are lengths and scale with δ by hypothesis. A common change of pace changes when births, deaths and pushes happen and not which, since each is set per episode. In the Gibbs world of FR16 Definition 10 the tension energy reads only the foam, which the rescaling leaves unchanged, and with zδ^d fixed the Gibbs weight against the Poisson process is unchanged. So the rescaled process has the same law, and its density is the unit density divided by δ^d. ∎

What it means. Pace drops out of every reading a keeper makes where it stands. Whatever law ties a keeper's extent to its pace, no keeper can detect it from where it stands; Section 10 returns to this.

Setting. *FR16 Definition 10's tension world with every length a fixed multiple of an extent δ(x) that varies slowly: hardcore δ(m), cut $R_{cut}\,\delta(m)$ and interaction ranges scaled with δ, for a pair of keepers with midpoint m; depths, rest values, the cap M and λ unchanged; birth intensity $z(x) = z_0\delta(x)^{(-d)}$. It is run as FR16 B.3's births and deaths: every keeper dies at rate 1, and a keeper is born at x at rate $z(x)\cdot I(x, \omega)\cdot\exp(-\Delta H(x, \omega))$, with the hardcore and the energy read at these scales. Write $\ell_\delta = 1/\sup|\nabla \ln \delta|$, the length over which δ changes, and assume $\lambda < \lambda_0$ with FR16's λ_0 for the unit extent.*

Theorem LE. *For ℓ_δ large against the interaction range: (i) the births and deaths have a unique stationary state; (ii) near every x_0 it agrees with the frozen tension world at extent $\delta(x_0)$: for every event E decided by the keepers within $R\delta(x_0)$ of x_0, $|P(E) - P_{(x_0)}(E)| \le C_R\,\xi/\ell_\delta$, where $\xi = 2R_{int}\,\delta(x_0)\Sigma_n\, n\,\alpha^n$ and $\alpha \le 3/4$ is FR16 B.3's branching bound; (iii) the density of keepers at x is $n_1\delta(x)^{(-d)}(1 + O(\xi/\ell_\delta))$, where n_1 is the density at unit extent.*

Proof. (i) FR16 B.3's backward construction uses only that birth rates are bounded by a candidate rate and that a candidate's fate is decided by the candidates in a look ball of bounded radius. Here a candidate at x has look radius $2R_{int}\,\delta(x)$ and the candidates in that ball have intensity at most $z_0 e^{(C_\lambda)}\delta(y)^{(-d)}$, with $\delta(y)/\delta(x) = 1 + O(R_{int}/\ell_\delta)$, so the expected number of ancestors is $\alpha(1 + O(R_{int}/\ell_\delta)) < 1$. Clans are subcritical everywhere with exponential tails, and the construction gives the unique stationary state (Fernández, Ferrari and Garcia 2002).

(ii) Build both worlds, inside $B(x_0, \ell_\delta/C)$, from one candidate process of intensity $\max(z(x), z(x_0))e^{(C_\lambda)}$ with shared marks, each world keeping a candidate when its mark is below that world's birth rate over the candidate rate, and independently outside. A keeper within $R\delta(x_0)$ of x_0 can differ between the worlds only if some member of the clans of the candidates there has a test that differs, given that its ancestors agree, or the clans leave the ball, which has probability at most $C_R e^{(-c\ell_\delta/R_{int}\,\delta(x_0))}$. Given agreeing ancestors, a test at distance ℓ from x_0 differs only if (a) the mark falls between the two worlds' thresholds, whose ratio is $1 + O(\ell/\ell_\delta)$, with probability $O(\ell/\ell_\delta)$; or (b) the configuration it reads lies within a relative margin $O(\ell/\ell_\delta)$ of a threshold of the rule: a keeper within that margin of the hardcore distance, or a cell-face ball whose radius lies within it of the cut.

The kept keepers are among the candidates, a Poisson process, so by the Mecke equation each has probability O(ℓ/ℓ_δ) per look ball. The cells are Voronoi cells, which do not depend on δ at all, so when the ancestors agree the Delaunay simplices the rule reads, those of circumradius up to the cut, depend only on the keepers in the look ball and are the same in both worlds, and only the hardcore and cut margins can change the reach, a count read from the cut foam. Generation n of a clan lies within 2nR_int δ(x_0) of its root and has expected size at most α^n, so the expected number of differing tests is at most Σ_n α^n C(Rδ(x_0) + 2nR_int δ(x_0))/ℓ_δ ≤ C_R ξ/ℓ_δ.

(iii) Apply (ii) to the event that a small ball about x holds a keeper, and use Lemma 8.0: the frozen world at extent δ(x_0) is the unit world rescaled, with density exactly n_1δ(x_0)^(−d). ∎

Corollary B′ (tension worlds with round cells). *Let the extent be δ(x) = εδ̂(x) with δ̂ smooth, the cells the Voronoi cells of the tension world, and keeper k's clock of rate ε^(−1)N(x_k). Then Theorems B and O hold with g = −c_T²N² dt² + n_1^(2/d)δ̂(x)^(−2)δ_ij dx^i dx^j, where c_T is the front speed (Lemma A.3) of the tension world's Voronoi foam at unit density. These are the round cells of Section 1, with ℓ = n_1^(−1/d)δ̂.*

Proof. The frozen worlds are FR16 Theorem 6.1's worlds rescaled, which have W1, W2 and W3, so FR16's Appendix A applies to them, and Lemma 1.1 holds with c_T for c_0. Lemma D applies with the affected keepers those present in one world only, plus those whose Voronoi cells depend on such keepers. For Ž_Q take, as in FR16 B.4's proof of W3, the local event E_Q, or a differing keeper within 6sε of Q, or the event F_Q that some clan explored from the candidates near Q leaves a neighbourhood U_Q of fixed size. Ž_Q is decided by the candidate process inside U_Q, so cubes far apart are independent, and P(Ž_Q = 1) ≤ p(s, b) + C s^d C_R ξ/ℓ_δ + z̄(5s)^d exp(−γ(s − 2R_int)) by Theorem LE(ii) and FR16 B.3. Lemma U holds for the tension world with varying extent, since B.4's proofs use only domination by a Poisson process of bounded intensity, the bound on empty balls and the clan tails, all uniform here. Theorem B follows as in Section 5. In Theorem O the keepers are not Poisson, so Steps 2 and 4 use the Mecke equation on the clocks alone with FR16 Lemma A.5's grid of points and Lemma A.4, as FR16 A.5 does, with dominated convergence over grid points supplied by Lemma U's moments. ∎

What it means. In the tension world, extent that varies slowly makes density follow extent by a theorem, not by choosing the keepers' intensity, and the order of such a world reads the conformally flat g. Cells of varying shape in the tension world would need a tension that reads direction; that is not done here.

9. Matter, Its Own Time and Free Fall

In plain words. Keepers do not move, so moving matter is hold passed from keeper to keeper. A lump keeps its time by the order, not by the clocks of the keepers it happens to hold, and a free lump has exactly one way to step that favours no direction. That step is falling.

Definition 9.1 (Held matter). *A lump is a composite X (FR16 Definition 5) whose constituents are held keepers. A moving lump is hold passed between keepers; the held count is passed on and never made. Each held keeper carries one fixed amount σ_0 at cell scale ε, so the mass of a lump is σ_0 times its held count, and the matter of a region at large scale is dust of density ρ = σ_0n, with n the density of held keepers in the dust's own rest frame and u its four-velocity. In the keepers' frame the density of held keepers is nu^0, the same at first order in velocity.*

Definition 9.2 (A lump's own time). *For episodes e ≺ f, τ(e, f) = κ_d ε|I(e, f)|^(1/(d+1)), with |I| the number of episodes strictly between and κ_d, which depends on d and c_0, fixed so that in FR16's flat world τ tends to Minkowski proper time as the interval grows.*

Remark. By Theorem O(1), ε^(d+1)|I(e, f)| tends to the μ-volume of g's causal diamond from e to f, and a small diamond's volume is a fixed multiple of the (d + 1)-th power of its proper time, so at scales

large against a cell and small against the curvature τ(e, f) reads g's proper time between e and f, in every frame.

Proposition 9.3 (proper time adds exactly on geodesics). *For nearby events $p \prec q \prec r$ joined by timelike curves, $\tau(p, r) \geq \tau(p, q) + \tau(q, r)$, with equality exactly when q lies on the timelike geodesic from p to r. In flat spacetime, a step of own time Δ_0 at rapidity w followed by a step Δ at rapidity u has span squared $\Delta_0^2 + \Delta^2 + 2\Delta_0\Delta \cosh(u - w)$, which is least, $(\Delta_0 + \Delta)^2$, only at $u = w$.*

Proof. The inequality is the reverse triangle inequality of Lorentzian geometry, with equality only on a geodesic (O'Neill 1983, Chapter 5; Hawking and Ellis 1973, Section 4.5). In flat spacetime the span is the norm of the sum of the two step vectors, and the derivative of $\Delta_0^2 + \Delta^2 + 2\Delta_0\Delta \cosh(u - w)$ in u is $2\Delta_0\Delta \sinh(u - w)$, zero only at u = w, where the second derivative is positive. ∎

Conditions for a free lump. A lump X steps when its own interval holds n_0 episodes, so its consecutive episodes e_1, e_2, … are own time Δ apart. By (M) it carries structure m_X along its chain →X, and for a moving lump that includes its previous episode, and so the direction u of its last step, the unit timelike vector from e_(i−1) to e_i. FR16 notes that (M) is deterministic: carried structure is shaped by prior carries and pushes. X is free when nothing outside it bears on it directionally at its grain and it carries nothing that points beyond u.

Condition G (a composite acts at its own grain; FR16 Definition 5). *What is a test at the constituents' grain is carry at the composite's grain. So X's next step depends on m_X and on the order around e_i read at X's grain Δ, and not on the ticks of its individual constituents, which at that grain are its own internal carry.*

Condition S (smoothness). *The step depends Lipschitz-continuously on the local geometry rescaled by Δ.*

Theorem 9.4 (free fall is forced, under Conditions G and S). *Let a free lump's next episode e_(i+1), at own time Δ after e_i, be determined, or for a random rule distributed, by m_X and the order at grain Δ, under Conditions G and S. (i) In flat spacetime, e_(i+1) = e_i + Δu: the lump carries straight on, and its proper time adds exactly. (ii) In any smooth spacetime, with L the curvature length, take first ε/Δ → 0, so that the order read at grain Δ is that of g, and then Δ/L → 0 at fixed proper time T: the lump's path converges to the timelike geodesic of g with the same initial direction. (iii) For a random rule, the mean step is along u and the remainder has mean zero.*

Proof. (i) By Theorem O, in the limit ε/Δ → 0, the order read at grain Δ is that of g, and in a flat region none of Theorem O's readings depends on motion through the keepers (FR16 Theorem 5.10, item 3). So the order there has the symmetries of Minkowski spacetime, and a rule that depends only on it and on m_X must commute with every Lorentz transformation that fixes its data: such a transformation carries the order to one with the same law and leaves e_(i−1) and e_i where they are, so it must leave the chosen step (for a random rule, its law) unchanged. The candidates at own time Δ after e_i are the points $e_i + \Delta(\cosh\eta\, u + \sinh\eta\, n)$, with n a unit spatial vector in u's rest frame. The transformations fixing e_(i−1) and e_i include every rotation about u, and for $d \geq 2$ no non-zero spatial vector is fixed by all of them. So $\sinh\eta = 0$, and the one allowed step is $e_i + \Delta u$. By Proposition 9.3 it is the step at which the span from e_(i−1) is least, $\Delta_0 + \Delta$: proper time adds. (iii) An invariant law on the candidates has an invariant mean displacement, which lies along u by the same argument, and the deviations from it have mean zero. (ii) After the limit ε/Δ → 0 the step is a function of g near e_i alone. Rescaled by Δ about e_i, in normal coordinates at e_i, a smooth spacetime is flat up to terms of order $(\Delta/L)^2$. By Condition S the step differs from the flat-space step, which is the geodesic continuation, by order $\Delta(\Delta/L)^2$, a change of direction of order $(\Delta/L)^2$ per step. Over a fixed proper time T there are T/Δ steps, so the accumulated change of direction is of order $T\Delta/L^2$, which tends to 0 as $\Delta/L \to 0$ at fixed T and L. The limits are taken in this order: the cell small against the step, then the step small against the curvature. Broken geodesics whose kinks vanish in this way converge to the geodesic with the same initial data. ∎

Corollary 9.5 (proper time adds). *A free lump steps so that τ(e_(i−1), e_(i+1)) = τ(e_(i−1), e_i) + Δ, and its path is a geodesic of g. Free fall is the same for every lump, since nothing about a lump enters but its own episodes, and a moving lump's time runs slow, since its time is proper time.*

Remark (what a frame-free rule could add). Corrections to the straight step must be built from u and the curvature. The term linear in the Riemann tensor, R^a_bcd u^b u^c u^d, vanishes identically, so in empty space a correction needs a curvature gradient or square, and inside matter the Ricci tensor; each is of order (Δ/L)² per step or smaller and vanishes in the limit. Under (iii) a finite lump reads proper time from finite counts and swerves about the straight step with mean zero; swerves of this kind are known in causal sets (Dowker, Henson and Sorkin 2004). Their size needs the cell size and n_0 and is not fixed here. The mean in (iii) is over the rule's chance at each step. Part (ii) is proved for a determined step; for a rule with chance in it the path follows the geodesic in the limit only if the swerves vanish there, which their size decides.

What the conditions exclude. The definition of free and Conditions G and S are the hypotheses of Theorem 9.4, and forced means forced given them. A rule that hands hold over at a constituent's own tick, or counts episodes on the constituents' clocks, reads the keepers' frame: their clocks tick at rest in it. Condition G excludes such a rule, and Theorem 9.4 shows why it must: if the keepers' four-velocity were part of a lump's data at its own grain, the rotations fixing that data would be fewer, the allowed steps would form a one-parameter family toward or away from the keepers' frame, and matter would feel a preferred-frame force. A lump in a pressure gradient is not free: the gradient is a second direction, the allowed steps form a family in the plane of u and the gradient, and conservation fixes the turn. This record treats dust and does not take that up.

Proposition 9.6 (dust is conserved). *For dust of Definition 9.1 moving by Corollary 9.5, T^ab = ρu^a u^b satisfies ∇_a T^ab = 0.*

Proof. ∇_a(ρu^a u^b) = u^b ∇_a(ρu^a) + ρu^a ∇_a u^b. The first term is zero because the held count is passed on and never made, so ρu^a is a conserved current; the second is zero because each lump moves on a geodesic. ∎

10. Carrying Gravity: The Cone Reading

In plain words. Each keeper looks back along its own past light cone and finds the latest episode of every keeper it can see. Those records pass from neighbour to neighbour, so nothing jumps across space, and they move at exactly the push speed. Read with the one weight that no motion can detect, they carry matter's influence as a retarded field. Frame-freeness and Newton's law then fix how the reading sets clocks and cells, up to one ratio; one premise fixes that ratio.

Proposition 10.1 (the order's own past cone). *For an episode e and a keeper k′, let L_k′(e) be the latest episode of k′ in e's past.*

(i) Local rule. *L_k′(e) is the latest, along k′'s own episodes, of L_k′(p) over the immediate predecessors p of e (its carry predecessor and the episodes that push it), with L_k′(p) = p when p is an episode of k′.*

(ii) On the cone. *The set {L_k′(e)} holds exactly one episode for every keeper with an episode in e's past, and in a world of Definition 1 it lies, at large scale, on g's past light cone of e to within a vanishing fraction of the distance.*

(iii) The weight. *With weight 1/r_k′, r_k′ the keepers'-frame distance to k′, the readings Σ_k′ q(L_k′(e))/r_k′ of a quantity q held at the sources converge, suitably scaled, to n∫ q d^dx/\|x\| over the past cone, and that measure is the same in every frame. It is the only such measure, up to a constant.*

(iv) The wave. *In three space dimensions the reading of a source q is the retarded solution of $\Box\Phi = -4\pi nq$, and so travels at exactly the push speed c_0. In two space dimensions it is not a solution of a wave equation.*

Proof. (i) By FR16 Definition 3, an episode of k′ precedes e exactly when a chain of carries and pushes runs from it to e, and every such chain ends in a link from an immediate predecessor of e. So the episodes of k′ in e's past are those of k′ in the pasts of e's immediate predecessors, together with e itself if e is k′'s; k′'s episodes are totally ordered by carry, so the latest of them is the latest of the predecessors' latest. (ii) Every keeper whose episodes reach e has a latest one, and only one. In a world of Definition 1 the past form of Theorem W puts the last tick of k′ in e's past at g's past light cone of e, up to o(1). (iii) The keepers have density n per unit volume of the keepers' frame, so the scaled sum is a Riemann sum for $n\int q\, d^dx/|x|$ over the cone. The past cone of e without its tip is a single orbit of the Lorentz group for $d \geq 2$, and $d^dx/|x|$ is the invariant measure on it; an invariant measure on a homogeneous space is unique up to a constant. (iv) In three space dimensions the retarded Green function of $\Box$ is $\delta(t - r)/r$, which lives on the cone alone with exactly the weight 1/r, so the reading is the retarded solution, at speed c_0. In two space dimensions the retarded Green function fills the inside of the cone. ∎

Definition 10.2 (the cone reading). *Each keeper reads matter from its own past light cone: the latest episode of every source keeper it can see, by the local rule of Proposition 10.1(i), with the weight of Proposition 10.1(iii). What a record holds of its source is the source's held count and, by (M), the direction u of its lump's last step, so the quantity read is matter's stress-energy $T_{ab} = \rho u_a u_b$ (Proposition 9.6), component by component. The records pass from neighbour to neighbour and forward only, since the order runs one way (FR16 Axiom 1), and the field so read is retarded.*

On this form. Four things are stated plainly. First, the records are functions of the order alone, and each is updated locally; only the bookkeeping is per source, and a keeper's record grows with the number of sources it can see. Second, the reading is the action-at-a-distance form of a field theory (Fokker 1929; Wheeler and Feynman 1945, 1949), equivalent to retarded fields at first order; causal-set dynamics is expected to be nonlocal at intermediate scales on other grounds (Sorkin 2009). Third, the reading sets the pace and cells of the keepers that read it, so here what keepers hold enters the order. FR16 Section 5 works with fixed clocks, where it does not, and its theorems are about that setting and are untouched. That Theorems B, W and O carry over to paces and cells so set is the bridge of Section 12. The alternative, a few numbers per keeper updated by averaging its neighbours, spreads at a speed set by its own coefficients, so matching it to c_0 would be a chosen constant; the cone reading inherits c_0 because it propagates by latest episodes, as the order does. Fourth, the weight uses $r_{k'}$, the keepers'-frame distance to the source. How a keeper reads $r_{k'}$ from its own records is not derived here: the weight is part of the adopted form, and its local reading is part of Target S (Section 12). At rest no distance is needed: Formal Record 18's static reading has each keeper average its neighbours. Beyond first order the weight cannot stay one over distance: read with the distances keepers measure, it puts a logarithm into a static field (Formal Record 18, Section 2). For a static field beyond first order, the graph's own spreading carries it (Formal Record 18); how the cone carries that spreading in time is open (Formal Record 18, Section 7). The cone reading is the first-order form of carrying.

Proposition 10.3 (the coupling is fixed up to one ratio). *Let the metric perturbation χ_{ab}, with $g = \eta + \chi$, be a linear reading, by Definition 10.2, of matter's stress-energy T_{ab}, the same in every frame. Then $\chi_{ab} = \alpha\, Rd[T_{ab}] + \beta\, \eta_{ab}\, Rd[T]$, with Rd the retarded 1/r reading and $T = \eta^{ab} T_{ab}$. For static dust, $\chi_{00} = (\alpha + \beta)\, Rd[\rho]$ and $\chi_{ij} = -\beta\, Rd[\rho]\, \delta_{ij}$. So with Newton's law, $\chi_{00} = 2U$, U the Newtonian potential, and $\chi_{ij} = 2\gamma U\, \delta_{ij}$, the ratio is $\beta/\alpha = -\gamma/(\gamma + 1)$, and $\gamma = 1$ exactly when $\beta = -\alpha/2$.*

Proof. The reading has one kernel, the weight of Proposition 10.1(iii), so the map from T to χ is a linear map from symmetric tensors to symmetric tensors that commutes with Lorentz transformations. Symmetric tensors split into their trace and their traceless part, each irreducible under the Lorentz group for $d \geq 2$, so by Schur's lemma the map is a multiple of the identity plus a multiple of the trace

projection: α T_ab + β η_ab T. For dust at rest, T_00 = ρ, T_ij = 0 and T = −ρ with η = diag(−1, 1, …, 1), which gives the components stated, and γ = χ_ij/χ_00 = −β/(α + β). ∎

Premise P (extent follows pace). *Where a keeper's clock runs at rate N, its extent is N times the extent where N = 1, at first order. Equivalently, the cells' metric is h = N^(−2)δ at first order in a static field.*

Anchor. FR16 Definition 4 (C) and Definition 8: varying a keeper's activity alters which elements fall within its boundary, and a held boundary has extent (FR15, H2). Premise P makes that quantitative: a keeper's activity sets its pace, and its extent follows. By Lemma 8.0 no keeper can detect it from where it stands.

Proposition 10.4 (what Premise P is). *(a) Premise P is independent of Sections 1 to 9, Proposition 10.1 and Proposition 10.3: for every k > 0 the law that extent is N^k times its value at N = 1 gives a world in which all of them hold, with γ = k. (b) In d = 3, at first order, outside matter, the following are equivalent: Premise P; γ = 1; the Ricci tensor of g vanishes in every direction, not only in time; the trace-reversed reading χ̄_ab = χ_ab − ½η_ab χ has zero divergence for every conserved source.*

Proof. (a) With extent N^k, h = N^(−2k)δ, and with N = 1 − U at first order, h_ij = (1 + 2kU)δ_ij: γ = k. Sections 1 to 8 hold for every smooth h and N with Definition 1's bounds, and Lemma 8.0 for every law tying extent to pace. Proposition 10.1 is about the order alone. Proposition 10.3 then gives β/α = −k/(k + 1), a consistent coupling. Theorem 9.4 and Proposition 9.6 hold in every smooth g. The scalar–tensor theory of Brans and Dicke (1961) is, at first order, the member k = (1 + ω)/(2 + ω), and violates none of these. (b) For $g_{00} = -(1 - 2U)$, g_ij = (1 + 2γU)δ_ij and $\nabla^2 U = 0$ outside matter, the first-order Ricci tensor is $R_{00} = -\nabla^2 U$ and R_ij = (1 − γ)∂_i∂_jU − γδ_ij∇^2U. Outside matter $R_{00} = 0$ for every γ, while R_ij = (1 − γ)∂_i∂_jU, which vanishes for all i and j exactly when γ = 1. For Proposition 10.3's reading, χ̄_ab = α Rd[T_ab] − (α/2 + β)η_ab Rd[T], whose divergence is −(α/2 + β)∂_b Rd[T] when ∂^a T_ab = 0; it vanishes for every conserved source exactly when β = −α/2, that is γ = 1. ∎

Remark (experiment). The radio link of the Cassini spacecraft measured $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ (Bertotti, Iess and Tortora 2003), so the exponent of Proposition 10.4(a) is 1 to that accuracy; Brans–Dicke needs ω above about 40,000 to pass. Premise P is the one place this record takes from nature what Einstein's field equations would otherwise supply. Formal Record 18 extends it to its exact four-dimensional form, the vanishing of the Ricci tensor on spatial directions outside matter, which agrees with it at first order (FR18 Proposition 4.1).

11. Gravity Where It Is Weak

In plain words. Put the pieces together. The cone reading of dust sets each keeper's pace and the size and shape of its cell; Theorems B, W and O prove that clocks and cells given in advance make exactly the curved spacetime they describe, and Theorem 11.1 is stated where that holds for the clocks and cells the reading sets (Section 12); and free lumps fall along its geodesics. What comes out, at first order, is Einstein's gravity.

Theorem 11.1 (weak-field gravity from held edges). *Work in d = 3, in units with $c_0 = 1$. Let matter be dust of Definition 9.1, and let keepers' paces and cells be set by the cone reading (Definition 10.2) with Premise P: the reading of a lump's held count sets the pace N = 1 − U, the extent follows by Premise P, and the traceless part of the reading of its stress sets the cells' shape. Then, at first order in U and in velocity, with G the reading's one constant:*

$$\bar{\chi}_{ab} = 4G\,\mathrm{Rd}[T_{ab}], \quad \chi_{ab} = 4G\,(\mathrm{Rd}[T_{ab}] - \tfrac{1}{2}\,\eta_{ab}\,\mathrm{Rd}[T]),$$

Keepers stay where they are and the size of their cells is fixed in time (Definition 1), so they hold the form of χ with zero shift and fixed cell size, where the first-order freedom of coordinates gives one:

for static sources; for the transverse traceless waves, χ^TT; and for stationary sources whose shift has zero divergence (Proposition 11.3). Where Theorems B, W and O hold for the paces and cells so set (the bridge, Section 12), the order of the world is that of g = η + χ at large scale, and:

(i) Newton's law. *A lump released at rest accelerates toward the source at ∇U, with U = G∫ρ(x′)/|x − x′| d³x′, the same for every lump.*

(ii) Light. *Push fronts passing a mass M at distance b turn through 2(1 + γ)GM/b = 4GM/b, and are delayed by Shapiro's 2(1 + γ)GM log term, with γ = 1.*

(iii) Waves. *The transverse traceless part χ^TT_ij = 4G Rd[T_ij]^TT solves □χ^TT = −16πG T^TT and travels at exactly c_0, with two polarizations and no scalar part: cells that hold it make the wave, and the local rule by which keepers come to hold it is Target S (Section 12).*

(iv) Frame dragging. *A stationary spinning source of angular momentum J drags the orbits of free lumps by the Lense–Thirring field (G/r³)[3(J·n)n − J] (Lense and Thirring 1918; Propositions 11.2 to 11.4).*

Proof. By Proposition 9.6 the dust is a conserved source, so Proposition 10.3 applies, and Premise P with Proposition 10.4 gives β = −α/2. Newton's law fixes α: χ_00 = (α/2)Rd[ρ] = 2U with U = G Rd[ρ], so α = 4G, which gives the display. Static dust: χ_00 = 2U and χ_ij = 2Uδ_ij, so g = −(1 − 2U)dt² + (1 + 2U)δ_ij dx^i dx^j. The pace N = 1 − U sets g_{00} at first order; Premise P sets the cells' size, h = N^(−2)δ = (1 + 2U)δ at first order; Theorem B makes the order of such a world that of g, and Theorem O its proper times. (i) By Corollary 9.5 a free lump follows a geodesic of g, whose slow-motion form is d²x/dt² = ∇U. (ii) Null geodesics of a metric with g_{00} = −(1 − 2U) and g_ij = (1 + 2γU)δ_ij turn through 2(1 + γ)GM/b and are delayed by the factor (1 + γ) on the Newtonian term (Will 2014); by Theorem B the push fronts of the order follow g's first arrivals. (iii) χ^TT has zero shift and is traceless, so cells that hold it change shape at fixed size, which is Theorem W's setting; Proposition 10.1(iv) gives the retarded solution at speed c_0, and by Proposition 10.4(b) there is no scalar part. The transverse traceless part of a field is not in general a local function of it, so this proves what the cells must hold and that, holding it, they make the wave; the local rule by which each keeper sets its cell's shape from its own records is Target S (Section 12). (iv) Propositions 11.2 to 11.4. ∎

Proposition 11.2 (the dragging coefficient is forced). *For dust moving at velocity v, the coupling of Theorem 11.1 gives $\chi_{0i} = -4V_i$, with V_i = G Rd[ρv_i], Einstein's value in harmonic gauge: in post-Newtonian terms 2γ + 2 = 4.*

Proof. At first order in v, $T_{0i} = -\rho v_i$, and $\eta_{0i} = 0$, so $\chi_{0i} = 4G\,\mathrm{Rd}[T_{0i}] = -4G\,\mathrm{Rd}[\rho v_i]$. A lump's velocity is part of what it carries under (M) (Section 9), so each held keeper's record carries its lump's velocity. ∎

Proposition 11.3 (keepers at rest can hold it). *For a stationary spinning source, $\chi_0 = -2G\,J \times r/r^3$ has zero divergence. The change of coordinates $\xi_i = \int \chi_{0i}\,dt$ removes the shift and moves it into the cells, $\chi'_{ij} = \chi_{ij} - t(\partial_i\chi_{0j} + \partial_j\chi_{0i})$: a shear that grows steadily, at fixed cell size, with the linearized curvature unchanged.*

Proof. $\nabla\cdot(J \times r/r^3) = r\cdot(\nabla \times J)/r^3 - J\cdot(\nabla \times (r/r^3)) = 0$, since J is constant and r/r^3 is a gradient. The trace of the added term is $-2t\ \nabla\cdot\chi_0 = 0$, so the cells' size is unchanged; the change is a coordinate change, so the linearized Riemann tensor is unchanged. The result has zero shift, shape changing in time at fixed size and clocks given, which is Theorem W's setting. ∎

Proposition 11.4 (the Lense–Thirring field). *With $g_i = \chi_{0i}$, the frame-dragging field $\Omega = -\tfrac{1}{2}\nabla \times g$ equals (G/r³)[3(J·n)n − J], and free lumps, moving on geodesics (Corollary 9.5), have their orbits dragged as Einstein's theory says.*

Proof. For $g = -2G\,J \times r/r^3$, $\nabla \times (J \times r/r^3) = J(\nabla\cdot(r/r^3)) - (J\cdot\nabla)(r/r^3) = -(J\cdot\nabla)(r/r^3)$ away from the origin, and $(J\cdot\nabla)(r/r^3) = J/r^3 - 3(J\cdot r)r/r^5$. So $\Omega = -\tfrac{1}{2}(-2G)(-(J/r^3 - 3(J\cdot n)n/r^3)) = (G/r^3)[3(J\cdot n)n - J]$. ∎

What it means. At first order, gravity is not added to the spacetime of held edges. Clocks that run slow where the reading of matter is large, cells that follow them in size and take their shape from the reading of stress, and lumps that carry straight on make exactly what Einstein's theory makes where gravity is weak, for the sources Theorem 11.1 covers, with one constant, one premise and the adopted form of reading, once the bridge of Section 12 holds. The orbits of the LAGEOS satellites measure this dragging (Ciufolini and Pavlis 2004), and gravitational waves have been seen to travel at the speed of light to about one part in 10^{15} (LIGO Scientific and Virgo Collaborations et al. 2017).

12. Scope, Target, Two Tiers and Falsification

What this record does not cover.

- Beyond first order: gravity acting on itself, the exact field of a lump and the classic tests at second order are not reached here; Formal Record 18 takes up the static field outside a held body.
- Pressure and stress: matter is dust. A lump in a pressure gradient is turned by conservation, but a law of stress for held keepers is not given here.
- Sources whose Newtonian field changes in time change the cells' size, which needs keepers that are born and leave. The cone reading gives the first-order field of such sources, but Theorems B, W and O are proved for fixed size only.
- Frame dragging held as shear works while the shear stays small, for times short against $1/(GJ/c^2r^3)$, about 1.9 million years at the Earth's surface; beyond that something else must hold it. Gyroscopes need a rule for how a lump carries orientation, which Theorem 9.4 does not supply.
- How the constituents of a lump carry out its step (the local hand-off) is open; Theorem 9.4 fixes what the step must be. The size of the swerve, the step count n_0 and the cell size are not fixed.
- Also not covered: rotating black holes, interiors and horizons, the cosmological constant and space whose size changes in time.

Target S (the local rules). A rule by which each keeper sets the shape of its cell from its own records alone, so that in the wave zone the cells hold the transverse traceless part of the cone reading. Theorem 11.1(iii) fixes what the cells must hold, and that cells holding it make the wave at c_0; how keepers hold it locally is open. The same holds for the weight of Proposition 10.1(iii): how a keeper reads the distance r_k′ from its own records is open.

The bridge from the reading. Theorems B, W and O are proved for clock rates and cells given in advance, with clocks independent of the order. In Theorem 11.1 the cone reading sets them, so they are random and depend on the order's past. That the reading concentrates on its mean as $\varepsilon \to 0$, and that Theorems B, W and O hold for rates and cells so set, is not proved here; Theorem 11.1 is stated where the world lies within them.

The governance tier and the ontological tier. As in FR16 Section 7, the record speaks at two depths and they are kept apart. No theorem, definition or gate condition in FR01 to FR13 depends on it. Within the full corpus, the time and space in which a natural person declares authority are produced by held boundaries (FR16), and the gravity that holds her to the ground is, on this record's premise and reading, what those boundaries do where their clocks and cells vary.

Falsification conditions. Within the ontological tier, this record is falsified by any of the following:

- **Light.** A measurement of γ that departs from 1 beyond its errors. It falsifies Premise P and with it Theorem 11.1(ii).
- **Waves.** Gravitational waves whose speed differs from that of light, or a scalar or vector polarization in waves from a source that the first-order field covers.
- **Fall.** A difference in free fall between bodies that depends on what they are made of, or a force on matter that depends on its motion relative to a frame, at scales where Theorem O applies.
- **Proofs (a check on the proofs, not a test against nature).** A world of Definition 1 whose order does not converge at large scale to that of g on the readings of Theorems B, W and O.

Theorems B, W and O prove that convergence, so such a world would show an error in their proofs.

Core Finding

Gravity is not added to the spacetime that held edges make. Where keepers' clocks run slow and, by Premise P, their cells shrink with them, the order they make is curved. A free lump knows only its last step, and the only step that favours no direction is to carry straight on: in curved spacetime, that is falling. Each keeper reads matter from its own past, and the reading moves at the push speed with nothing tuned. Given the one premise and the adopted reading, and once the bridge of Section 12 holds, what results where gravity is weak is Einstein's gravity at first order. Falling is carrying on.

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